Last Time

- When program S executes it switches to a different state
- We need to express assertions on the states of the program S before and after its execution
- We can do it using a Hoare triple written as {P}S{Q}, where P is a precondition, S is a program, and Q is a postcondition
- We used flowchart diagrams to prove partial correctness and termination of two programs

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Inference Rules

- An inference rule maps one or more wffs, called premises, to a single wff, called the conclusion

$$\frac{A, A \to B}{\therefore B} \text{ modus ponens (MP)} \qquad \frac{A \vee B, \neg A}{\therefore B} \text{ disjunctive syllogism (DS)}$$

$$\frac{\neg B, A \to B}{\therefore \neg A} \text{ modus tollens (MT)} \qquad \frac{A \to B, B \to C}{\therefore A \to C} \text{ hypothetical syllogism (HS)}$$

$$\frac{A, B}{\therefore A \wedge B} \text{ conjunction intro (CI)} \qquad \frac{A \vee B, A \to C, B \to D}{\therefore C \vee D} \text{ constructive dilemma (CD)}$$

$$\frac{A}{\therefore A \vee B} \text{ disjunction intro (DI)} \qquad \frac{\neg C \vee \neg D, A \to C, B \to D}{\therefore \neg A \vee \neg B} \text{ destructive dilemma (DD)}$$

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Proofs

- A proof is a finite sequence of wffs s.t. each wff in the sequence is either an axiom or a premise or can be inferred from previous wffs in the sequence
- A formal reasoning system is also called a formal theory
- If a formal theory enables the proof of both wffs P and ¬P, then this theory is inconsistent (not sound)
- How to build consistent theories?
  - Choose axioms to be tautologies
  - Choose inference rules to map tautologies onto tautologies
- Examples
  - Prove $(A \vee B) \wedge (A \vee C) \wedge \neg A \rightarrow B \wedge C$
    1. $A \vee B$ P
    2. $A \vee C$ P
    3. $\neg A$ P
    4. B 1,3,DS
    5. C 2,3,DS
    6. $B \wedge C$ 4,5,CI
    7. QED 1,2,3,6

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Our Strategy

- Recall proof calculi for propositional and predicate logic
  - Formula to prove, inference rules, axioms
  - For example, to prove $\phi \rightarrow \varphi$ we assume ϕ and manage to show φ using given inference rules
- What if we replace a logic formula with a piece of code?
- Can we prove fragments of code and these small proofs compose a final proof?

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Partial Correctness, Termination, and Total Correctness

- **Partial correctness**: if for all states that satisfy the precondition, the state resulting from program's execution satisfies the postcondition, provided that the program terminates
- **Termination**: if the precondition holds, then the program terminates
- **Total correctness**: if for all states in which P is executed which satisfy the precondition, P is guaranteed to terminate and the resulting state satisfies the postcondition

Proof Calculus For Partial Correctness

- Goes back to R.Floyd and C.A.R. Hoare
- Given a language grammar
- Given proof rules for each of the grammar clauses for commands
- We construct our proofs in a form of proof tableaux

A Core Programming Language

- S ::=
  x=E |
  S;S |
  if B {S} else {S} |
  while B {S}
- B ::= true | false | (!B) | (B&B) | (B||B) | (E<E)
- E ::= n | x | (-E) | (E-E) | (E+E) | (E*E)
- n is any numeral
- x is any variable

A Program For Computing a Factorial

```
Factorial( x ) {
  y = 1;
  z = 0;
  while( z != x) {
      z = z + 1;
      y = y * z;
  }
}
```

$$0! \triangleq 1$$

$$(n+1)! \triangleq (n+1) \cdot n!$$

Composition Rule

$$\frac{\{P\}\, S_1\, \{Q\} \qquad \{Q\}\, S_2\, \{R\}}{\{P\}\, S_1; S_2\, \{R\}}$$

- S_1 and S_2 are program fragments
- In order to prove {P} S_1;S_2{R} we need to find an appropriate Q
- Then we prove {P} S_1{Q} and {Q}S_2{R} separately

Assignment

$$\frac{}{\{P[E/x]\}\, x = E\, \{P\}}$$

- No premises => it is an axiom!
- We wish to know that P holds in the state after the assignment x = E
- P[E/x] means the formula obtained by taking P and replacing all occurrences of x with E
  - P with E **in place of** x

Assignment: Flawed Understanding

$$\frac{}{\left\{P\left[E/x\right]\right\}\; x = E \left\{P\right\}}$$

- If P holds in a state in which we perform the assignment x = E, then P[E/x] holds in the resulting state
  - We replace x by E
  - Do we perform this replacement of occurrences of x **in a condition** on the starting state by E?

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Assignment: Correct Understanding

$$\frac{}{\left\{P\left[E/x\right]\right\}\; x = E \left\{P\right\}}$$

- Do we perform this replacement of occurrences of x **in a condition** on the starting state by E?
- No, we need to prove something about the initial state in order to prove that P holds in the resulting state
- Whatever P says about x but applied to E must be true in the initial state

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Assignment: Examples

$$\overline{\{2=2\}\,x=2\,\{x=2\}}$$

- If we want to prove x=2 after the assignment x=2, then we must be able to prove that 2=2 before it

$$\overline{\{2=y\}\,x=2\,\{x=y\}}$$

- If we want to prove x=y after the assignment x=2, then we must be able to prove that 2=y before it

Assignment: Exercises

$$\overline{\{x+1=2\}\,x=x+1\{x=2\}}$$

$$\overline{\{x+1=y\}\,x=x+1\{x=y\}}$$

$$\overline{\{x+1>0\wedge y>0\}\,x=x+1\{x>0\wedge y>0\}}$$

Assignment

$$\overline{\{P[E/x]\}\ x = E\ \{P\}}$$

- This assignment axiom is best applied backward than forward in the verification process
- We know Q and wish to find P s.t. {P}x=E {Q} – easy
  - Set P to be Q[E/x]
- If we know P and want to find Q s.t. {P} x=E {Q} – very difficult!!!

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IF-Statement Rule

$$\frac{\{P \wedge B\}\ S_1\ \{Q\} \qquad \{P \wedge \neg B\}\ S_2\ \{Q\}}{\{P\}\ \text{if}\ \ B\ \ \{S_1\}\ \text{else}\ \{S_2\}\ \{Q\}}$$

- S_1 and S_2 are program fragments
- Decompose the if rule into two triples
- Then we prove these triples separately

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WHILE-Statement Rule

$$\frac{\{P \wedge B\}\, S\, \{P\}}{\{P\}\ \text{while } B\ \{S\}\{P \wedge \neg B\}}$$

- S is a program fragment that is executed multiple times in the while loop
- We don't know how many times S is gonna be executed or whether it terminates at all
- P is a loop invariant

Implied Rule

$$\frac{\vdash \mathrm{P}' \rightarrow \mathrm{P} \qquad \{\mathrm{P}\}S\{Q\} \qquad \vdash \mathrm{Q} \rightarrow \mathrm{Q}'}{\{\mathrm{P}'\}S\{Q'\}}$$

- Implied rule allows the precondition to be strengthened
  - We assume more than we need to
- The postcondition may be weakened
  - We conclude less than we are entitled to

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```
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        y = y * z;
   }
}
```

$$0! \triangleq 1$$

$$(n+1)! \triangleq (n+1) \cdot n!$$

Let's Prove It!!!

Proof Tableaux

- What is good about them?
  - Tree structure
  - We think of a program as a sequence of code fragments
- We interleave the program code with intermediate formulae called midconditions
- Is it easy to read proof tableaux?
- Is there an alternative?

Division With Remainder Example

$\{x \geq 0 \wedge y \geq 0\}$

$a = 0;$
$b = x;$
while $(b \geq y)$ {
  $b = b - y;$
  $a = a + 1;$
}

$\{x = a \cdot y + b \wedge b \geq 0 \wedge b < y\}$

DivProg

Invariant:

$$\{x = a \cdot y + b \wedge b \geq 0\}$$

Invariant

- How to start the proof?
- Heuristics: Find invariant for each loop.

- For this example: x=a*y+b ∧ x>=0
- Note: total correctness does not hold for y=0
- Total correctness (with y>0) should be proved separately.

Proof

$$\{x = a \cdot y + x \wedge x \geq 0\}\, b = x\, \{x = a \cdot y + b \wedge b \geq 0\}$$ 1

$$\{x = 0 \cdot y + x \wedge x \geq 0\}\, a = 0\, \{x = a \cdot y + x \wedge x \geq 0\}$$ 2

$$\{x = 0 \cdot y + x \wedge x \geq 0\}\, a = 0; b = x\, \{x = a \cdot y + b \wedge x \geq 0\}$$ 3

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Proof

$$\{x = (a+1) \cdot y + b \wedge b \geq 0\}\, a = a + 1\, \{x = a \cdot y + b \wedge b \geq 0\}$$ 4

$$\{x = (a+1) \cdot y + b - y \wedge b - y \geq 0\}\, b = b - y$$
$$\{x = (a+1) \cdot y + b \wedge b \geq 0\}$$ 5

$$\{x = (a+1) \cdot y + b - y \wedge b - y \geq 0\}\, b = b - y; a = a + 1$$
$$\{x = a \cdot y + b \wedge b \geq 0\}$$ 6

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Consequence rules

- Strengthen a precondition

$$\frac{R \rightarrow P \qquad \{P\}\,S\,\{Q\}}{\{R\}\,S\,\{Q\}}$$

- Weaken a postcondition

$$\frac{\{P\}\,S\,\{Q\} \qquad Q \rightarrow R}{\{P\}\,S\,\{R\}}$$

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Proof

$$(x=a\cdot y+b\wedge b\geq 0\wedge b\geq y)\rightarrow(x=(a+1)\cdot y+b-y\wedge b-y\geq 0)$$ 7

$$\{x=a\cdot y+b\wedge b\geq 0\wedge b\geq y\}\,b=b-y;a=a+1$$

$$\{x=a\cdot y+b\wedge b\geq 0\}$$ 8

consequence, 6, 7

$$\{x=a\cdot y+b\wedge b\geq 0\}\,\text{while }(b\geq y)\ \{$$

$$b=b-y;a=a+1$$

$$\{x=a\cdot y+b\wedge b\geq 0\wedge b<y\}$$ 9

while, 8

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Proof

$\{x = 0 \cdot y + x \wedge x \geq 0\}$ DivProg
$\{x = a \cdot y + b \wedge b \geq 0 \wedge b < y\}$
composition, 3,9

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$(x \geq 0 \wedge y \geq 0) \rightarrow (x = 0 \cdot y + x \wedge x \geq 0)$

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$\{x = 0 \cdot y + x \wedge x \geq 0\}$ DivProg
$\{x = a \cdot y + b \wedge b \geq 0 \wedge b < y\}$
consequence

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Soundness

- Hoare logic is sound in the sense that everything that can be proved is correct!
- This follows from the fact that each axiom and proof rule preserves soundness

Completeness

- A proof system is called complete if every correct assertion can be proved
- Propositional logic is complete
- No deductive system for the standard arithmetic can be complete (Godel)

And for Hoare's logic?

- Let S be a program and P its precondition
- Then {P} S {⊥} means that S never terminates when started from P
  - This is undecidable
  - Thus, Hoare's logic cannot be complete

General Observations

- If we can prove programs then we represent them as mathematical objects
- Does it mean that computer programmers are like mathematicians?
- Mathematicians try to improve their confidence in the correctness of theorems
- They use chain of formal logic statements to achieve this goal

Is Proof = Program?

- By verifying a program we increase our confidence in it
- So, it is like verifying the correctness of a theorem, right?
- The critical piece here is a social process that governs the acceptance of a theorem
- It is completely different between mathematical theorems and verified program

Mathematical Process

- Mathematicians publish about 200,000 theorems each year
- Are all of them correct and/or accepted?
- Multiple examples of famous mathematicians who announced and published proofs of theorems that were discredited later
  - Sometimes after many, many years!
- Mathematicians make a lot of mistakes!

Who Corrects Those Mistakes?

- Examples of contradictory results from published complicated proofs are well-known
- Only mathematicians can correct their errors, but who verifies the correctness of corrections?

- **A proof does not in itself significantly raise our confidence in the probable truth of the theorem it purports to prove**

What About Algebraic Proof?

- Many examples confirm that proofs that consist solely of calculations are not necessarily correct
- It is not the question of "how do theorems get believed?"
- It is a question of "what is it we believe when we believe a theorem?"

Long Proofs

- Given a proof that occupies 2,000 pages, how long would it take to verify its correctness?
- What is a value of a long and complicated proof?
- How social process works for mathematicians?
- What is a fundamental difference between mathematicians and computer scientists doing proofs?

Why Do We Need Program Verification?

- Testing can never show the absence of errors, only their presence
- Software errors can cause major disasters especially in critical systems
- Math is used to state program properties and to prove program correct for all inputs
- However, program verification is expensive and has other drawbacks

Man and Machines

- What parts of program verifications cannot be replaced by machines?
- How to choose what properties to prove?
- How to find errors in specifications?
- Is the proof process correct?

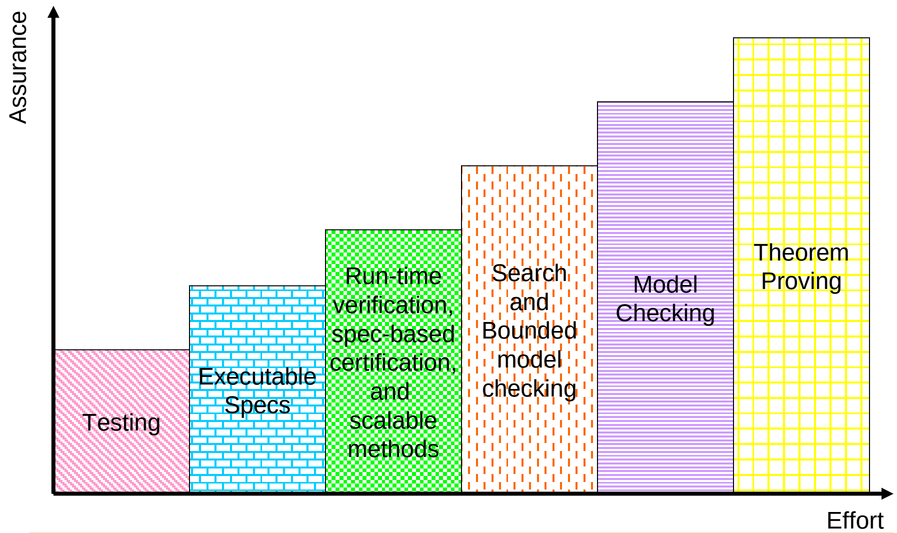
Tool-Assisted Verification

- We can use tools that mechanize the deduction process
- If we have executable specifications then we can use tools that assist us in debugging these specifications
- When doing proof of program correctness we can use theorem provers to ensure proof correctness

Limitations of Program Verification

- We have only limited ways to convince ourselves that we are given a correct spec
- Even with the right specification we can prove only the correctness of mathematical abstraction, never of the system running in the real world
- There is a significant cost associated with program correctness proofs
- Not all systems are equally critical

Cost and Assurance of Formal Methods



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Believing Software

- People cannot create perfect mechanisms
- Use social processes to create reliable structures
  - This is what most engineers do
- Computing structures are not
  - Perfect
  - The energy that can be wasted to make them perfect, is limited

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Homework

- **Mandatory**
  - R. De Millo, R. Lipton, and A. Perlis. "Social processes and proofs of theorems and programs," *Communications of the ACM*, 22(5):271-280, May 1979
  - R. Floyd, `Assigning meaning to programs', Proc. Symposium on Applied Mathematics, American Mathematical Society, 1967, Vol. 1, pp. 19--32.
  - J. Fetzer. "Program verification: The very Idea," *Communications of the ACM*, Vol. 31. No. 9. pp. 1049-1063.
    - Downloadable from http://www.swt.edu/~mg43/reading.html
- **Optional**
  1. Michael Huth and Mark Ryan, *Logic in Computer Science: Modelling and Reasoning about Systems*, Cambridge University Press, November 1999.

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