Maximum Necessary Hop Count for Packet Routing in MANET

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Contents

Chapter 1

Maximum Necessary Hop Count for Packet Routing in MANET

This paper investigates a fundamental characteristic of a mobile ad-hoc network (MANET): the maximum necessary number of hops needed to deliver a packet from a source to a destination. In this paper, without loss of generality, we assume that the area is a circle with a radius of r, $r > 1$, and the transmission range of each mobile station is 1. We prove that the maximum necessary number of hops needed to deliver a packet from a source to a destination is $\frac{4\pi}{\sqrt{3}}\left(r + \frac{1}{\sqrt{3}}\right)^2 - 1 = \frac{4\pi r^2}{\sqrt{3}} + O(r) \approx 7.255r^2 + O(r)$. We show that this result is very close to optimum with only a difference of $O(r)$.

Keywords: ad-hoc, mobile station, packet, transmission range, wireless network

1.1 Introduction

Recent advances in technology have provided portable computers with wireless interfaces that allow networked communication among mobile users. The resulting environment no

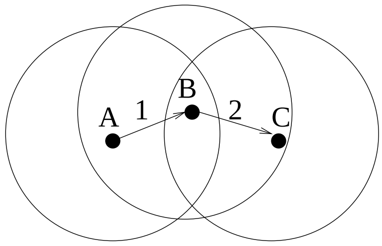

Figure 1.1: An example of an ad-hoc wireless network

longer requires users to maintain a fixed and universally known position in the network and enables almost unrestricted mobility.

A *mobile ad-hoc network (MANET)* is formed by a cluster of mobile stations randomly located within a certain area without the infrastructure of base stations. The applications of MANETs appear in places where predeployment of network infrastructure is difficult or unavailable, for example, fleets in oceans, armies in march, natural disasters, battle field, festival field grounds, and historic sites.

In a MANET, stations communicate with each other by sending and receiving packets. The delivery of a packet from a source station to a destination station is called *routing*. Particularly in a MANET, two stations can communicate directly with each other if and only if they are within each other's *wireless transmission range*. Otherwise, the communication between them has to rely on other stations. For example, in the network shown in Figure 1.1, stations A and B are within each other's transmission range (indicated by the circles around A and B respectively). If A wants to send a packet m to B, A can send it directly in one hop. A and C are not within each other's transmission range. If A wants to send a packet to C, it has to first forward the packet to B and then use B to route the packet to C. Therefore, it takes two hops to deliver a packet from A to C.

Many routing algorithms have been designed in the literature ([1], [3], [5], [6], [8]), but less work has been done to investigate fundamental properties of MANET in a mathematical way. In this paper, we address the issue of finding the maximum necessary number of hops needed to deliver a packet from a source to a destination in a MANET within a certain area. An initial attempt to solve the problem by us is made in [2]. In this paper, we will

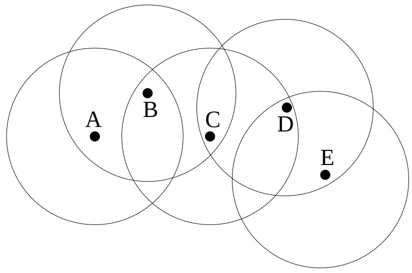

Figure 1.2: An example of an ad-hoc wireless network

reconsider the problem and provide a much better maximum necessary hop count. Without loss of generality, we assume that all the mobile stations are located within an area of a circle with a radius r, $r > 1$, and the transmission range of all the nodes is 1 assuming they use the fixed transmission power.

This paper is organized as follows: Section 1.2 is the notations. Section 1.3 puts forward the problem. Section 1.4 introduces the circle packing problem. Section 1.5 provides our solution to the problem. Section 1.6 shows the sharpness of our solution. And Section 1.7 is the conclusion.

1.2 Notations

We can use a simple graph $G = G(V, E)$ to represent a MANET, where the vertex set V is the collection of mobile stations within the wireless network. An edge between two stations u and v denoted by $u \leftrightarrow v$ means that both of them are within each other's transmission range. We assume that this graph G is finite and connected.

See an example of a wireless network in Figure 1.2. There are five stations A, B, C, D, and E in MANET. The circle around each one represents its transmission range. Two vertices are connected if and only if they are within each other's transmission range. The resultant graph is shown in Figure 1.3.

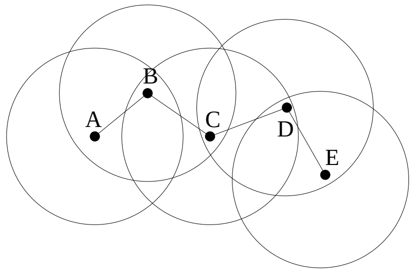

Figure 1.3: The graph that represents an ad-hoc wireless network

1.3 The Problem

In this paper, we address the issue of finding out the maximum necessary number of hops needed to deliver a packet from a source to a destination in a MANET within a certain area. Without loss of generality, we assume that all the mobile stations are within an area of a circle with a radius r, $r > 1$. Two stations u and v can communicate with each other if and only if their geographic distance is less than or equal to 1. Based on the above description, a simple graph G can be drawn to represent the MANET within this circle. The maximum necessary number of hops to deliver a packet from a source to a destination is actually the diameter of the graph G.

1.4 Circle Packing Problem

Before we present our result, we introduce the circle packing problem which leads to our solution.

The *circle/sphere packing* problem is to consider how to effectively pack non-overlapping small circles/spheres of the same size into a large circle/sphere as many as possible so that the *density of a packing*, which is the ratio of the region/space occupied by the circles/spheres to the whole region/space, is as large as possible.

The history of circle/sphere packing problem goes back to the early 1600s, when astronomer-mathematician Johannes Kepler asserted that no sphere packing could be better than the

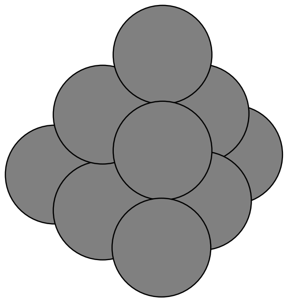

Figure 1.4: A pyramid in the Face Centered Cubic sphere packing

Face Centered Cubic (FFC) packing. The FFC packing is the natural one that arises from packing spheres in a pyramid, as shown in Figure 1.4.

The Kepler Conjecture *The density of any sphere packing in three-dimensional space is at most $\pi/\sqrt{18}$, which is the density of the FCC packing.*

Although the Kepler conjecture looks natural, it is very difficult to prove. Recently, a 250-page proof of the Kepler conjecture was claimed in [4].

Compared with the difficulties of the Kepler conjecture, the problem of circle packing in the plane has been solved with ease. The result is stated in the following lemma. It was first proved by Axel Thue in 1890. In order to make our paper self-contained, we include a proof provided in [7] with the following slight change: while the proof in [7] uses unit circles to pack the plane, we use circles of radius 1/2 to pack the plane in order to be consistent with the proof of Theorem 1 in Section 1.5.

Lemma 1 *The optimum packing of circles in the plane has density $\pi/\sqrt{12}$, which is that of the hexagonal packing shown in Figure 1.5.*

Proof. We may suppose without loss of generality that we use circles of radius 1/2 to pack the plane. Start with an arbitrary packing of circles of radius 1/2 in the plane. Around each circle draw a concentric circle of radius $1/\sqrt{3}$. (The reason to choose $1/\sqrt{3}$ as the radius of the concentric circle is when three 1/2 radius circles are packed next to each other just like

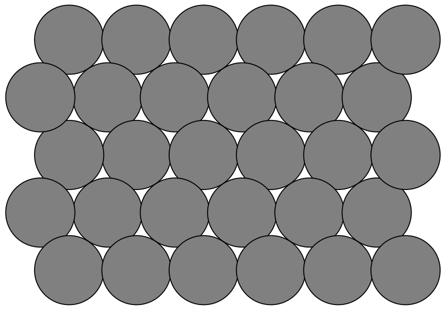

Figure 1.5: The hexagonal circle packing of the plane

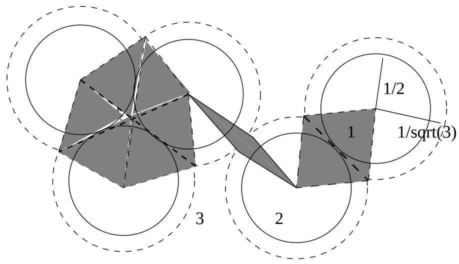

Figure 1.6: The plane partition induced by a sample circle packing

as shown in the left part of Figure 1.6, the three centers can form an equilateral triangle, then the three cooresponding concentric circles can intersect at the middle of the equilateral triangle.) Where a pair of concentric circles overlap, join their points of intersection and use this as the base of two isosceles triangles. The centers of each circle form the third vertex of each triangle. In this way, the plane can be partitioned into three regions, as depicted in Figure 1.6.

1. The isosceles triangles: if the top angle of the triangle is θ radians, then its area is $\left(1/\sqrt{3}\right)^2 \sin\theta/2 = \sin\theta/6$ and the area of the sector is $(1/2)^2\theta/2 = \theta/8$, so that the packing density is
$$\frac{\theta/8}{\sin\theta/6} = \frac{3\theta}{4\sin\theta},$$
where θ ranges between 0 and $\pi/3$ radians. The maximum value of the density is $\pi/\sqrt{12}$, attained at $\theta = \pi/3$ radians.

2. Regions of the larger circles not in a triangle: here the regions are sectors of a pair of

concentric circles of radius 1/2 and $1/\sqrt{3}$, so the density is

$$\left(\frac{1/2}{1/\sqrt{3}}\right)^2 = \frac{3}{4} < \frac{\pi}{\sqrt{12}}.$$

3. Regions not in any circle: here the density is zero.

Since the density of each region of space is at most $\pi/\sqrt{12}$, the density of this circle packing is at most $\pi/\sqrt{12}$. Furthermore, a packing can only be optimum when it causes space to be divided into equilateral triangles. This only happens for the hexagonal packing of Figure 1.5.

Remarks. The above lemma concerns circle packing into an unlimited space. Now consider that some circles C_i of radius 1/2 are packed into a limited region such as a large circle with a radius t. From now on, we use the notation $C(x)$ to represent a circle with a radius x. In the proof of Lemma 1, we draw a concentric circle $C(1/\sqrt{3})$ around each C_i. Since all circles C_i are within a circle $C(t)$, all those circles $C(1/\sqrt{3})$ must be within the circle $C(t + 1/\sqrt{3} - 1/2)$ which is concentric with $C(t)$. Thus the proof of Lemma 1 can still be applied to the limited region $C(t + 1/\sqrt{3} - 1/2)$; that is,

$$\frac{\sum_i (\text{area of } C_i)}{\text{area of } C(t + 1/\sqrt{3} - 1/2)} \leq \frac{\pi}{\sqrt{12}}.$$

Furthermore, Lemma 1 shows that $\pi/\sqrt{12}$ is the best upper bound for the ratio of the area of all C_i's to the area of $C(t + 1/\sqrt{3} - 1/2)$, in the case that $C(t)$ is sufficiently larger than each of the small circles C_i.

1.5 Our Solution

Based on the above lemma, the following is our result of the maximum necessary hop count to deliver a packet from a source to a destination in MANET.

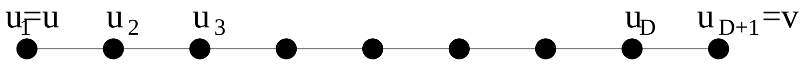

Figure 1.7: A shortest path between u and v

Theorem 1 *Assume that all the mobile stations are within an area of a circle with a radius r, $r > 1$. The transmission range of each mobile station is 1. Denote the graph generated by connecting all pairs of vertices within each other's transmission range as G; that is, two vertices are connected if and only if their geographic distance is less than or equal to 1. Then an upper bound for the diameter of G is $\frac{4\pi}{\sqrt{3}}\left(r+\frac{1}{\sqrt{3}}\right)^2-1$, in other words, it takes maximum $\frac{4\pi}{\sqrt{3}}\left(r+\frac{1}{\sqrt{3}}\right)^2-1=\frac{4\pi r^2}{\sqrt{3}}+O(r)$ necessary hops to deliver a packet from a source to a destination.*

Proof. Let D be the diameter of the graph G. Choose two vertices u, v such that the distance in G between u and v, $d_G(u,v)$, is D; that is, $d_G(u,v) = D$. Then there exist distinct vertices u_i, $1 \leq i \leq D+1$, such that

$$u = u_1 \leftrightarrow u_2 \leftrightarrow u_3 \leftrightarrow \cdots \leftrightarrow u_D \leftrightarrow u_{D+1} = v$$

is a shortest path of length D between u and v (see Figure 1.7).

We define a set $I = \{u_i : i \text{ is odd}\}$. Then the size of I, denoted by $|I|$, is $\lceil (D+1)/2 \rceil$. We can prove that I is an *independent set* of vertices. An independent set of vertices is defined as a set of vertices in which there is no edge between any pair of vertices in the set.

Claim 1: I is an independent set of vertices in graph G.

Proof of Claim 1. Suppose that I is not an independent set, that is, there exists at least one edge between some pair of vertices in I. Without loss of generality, we assume that there are two vertices u_{2j+1}, u_{2k+1} in I such that $j < k$ and $u_{2j+1} \leftrightarrow u_{2k+1}$. By the definition of I, the two vertices are on the shortest path from u to v. Then the shortest path can be represented as

$$u = u_1 \leftrightarrow \cdots \leftrightarrow u_{2j} \leftrightarrow u_{2j+1} \leftrightarrow u_{2k+1}$$

$$\leftrightarrow u_{2k+2} \leftrightarrow \cdots \leftrightarrow u_{D+1} = v$$

The length of this path, as represented, is $D - [(2k+1) - (2j+1)] + 1 = D + 2j - 2k + 1$. It is less than D. This contradicts to $d_G(u, v) = D$. Therefore there is no edge between any pair of vertices in I; in other words, I is an independent set in G.

By Claim 1 and the definition of G, we have the following claim.

Claim 2: The geographic distance between any pair of vertices in I is larger than 1.

Then, for each vertex $u_i \in I$, we define a circle S_i with a center at u_i and with a radius of 1/2. By Claim 2, two circles S_j, S_k are disjoint if $j \neq k$. Since all the vertices in I are covered by the circle $C(r)$, all the disjoint circles S_i (i is odd) can be covered by a larger circle named $C(r + 1/2)$ which has the same center as the circle $C(r)$ and has a radius of $r + 1/2$ (see Figure 1.8 for an example). Now we can relate this diameter problem to the circle packing problem, that is, how to effectively pack these non-overlapping circles S_i (i is odd) as many as possible into the larger circle $C(r + 1/2)$? By the remarks following the proof of Lemma 1 in Section 1.4 (Note that the circle $C(t+1/\sqrt{3}-1/2)$ in the remark should be converted to $C(r + 1/\sqrt{3})$ since $t = r + 1/2$.), we have

$$\frac{\sum_i (\text{area of } S_i)}{\text{area of } C(r + 1/\sqrt{3})} \leq \frac{\pi}{\sqrt{12}},$$

where $C(r+1/\sqrt{3})$ is the circle which has the same center as the circle $C(r)$ and has a radius of $r + 1/\sqrt{3}$. Thus

$$\frac{|I|\pi\left(\frac{1}{2}\right)^2}{\pi\left(r + \frac{1}{\sqrt{3}}\right)^2} \leq \frac{\pi}{\sqrt{12}}.$$

Solving for $|I|$,

$$|I| \leq \frac{2\pi}{\sqrt{3}}\left(r + \frac{1}{\sqrt{3}}\right)^2.$$

Since $|I| = \lceil (D+1)/2 \rceil \geq (D+1)/2$, we have

$$D \leq 2|I| - 1 \leq \frac{4\pi}{\sqrt{3}}\left(r + \frac{1}{\sqrt{3}}\right)^2 - 1.$$

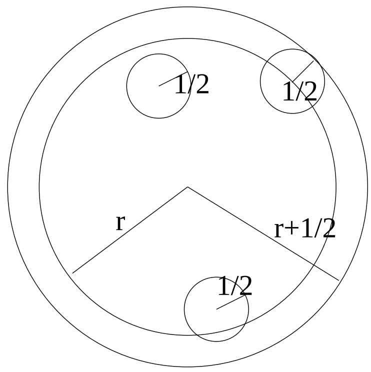

Figure 1.8: All the S_i's are covered by a circle of radius $r + 1/2$

So, it takes maximum $\frac{4\pi}{\sqrt{3}}\left(r+\frac{1}{\sqrt{3}}\right)^2 - 1 = \frac{4\pi r^2}{\sqrt{3}} + O(r)$ necessary hops to deliver a packet from a source to a destination.

1.6 Sharpness Of The Maximum Necessary Hop Count

In this section, we show the sharpness of the maximum necessary hop count. The idea is like this: if we can actually construct a graph G with a diameter of $4\pi r^2/\sqrt{3} + O(r)$ in a circle $C(r)$, then our maximum necessary hop count $\frac{4\pi}{\sqrt{3}}\left(r+\frac{1}{\sqrt{3}}\right)^2 - 1$ is very close to optimum with only a difference of $O(r)$. The construction of such a G is based on a parallelogram packing into the circle $C(r)$. Before the construction, we present some assumptions and properties of the packing unit "parallelogram".

Let ϵ $(0 < \epsilon < 1)$ be a positive real number. We draw a parallelogram $ABDE$. Then we choose points C and G on the side of BD, and choose a point F on the side of AE such that $|AC| = |AG| = 1 + \epsilon$, $|AF| = 1$, $|CG| = 1 - \epsilon$ and $|BG| = |CD| = |EF| = \epsilon$, where $|\ \ |$ represents the length of a line segment. (See Figure 1.9.) Then $|AB|$ and $|DE|$ can be uniquely determined in terms of ϵ since $ABDE$ is a parallelogram. Next we give some properties of the parallelogram.

Property 0: $|AF| = |BC| = 1$, $|EF| = |CD| = \epsilon$, and $|AE| = |BD| = 1 + \epsilon$.

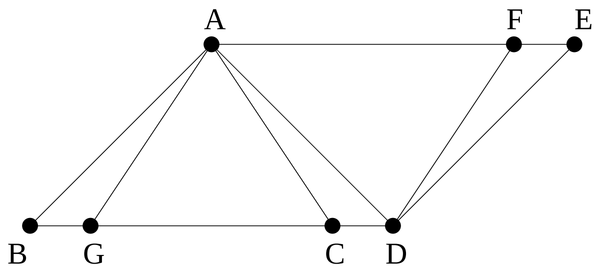

Figure 1.9: A parallelogram ABDE

This property is obvious from the above assumptions.

Property 1: Any two vertices chosen from sets $\{A, E, F\}$ and $\{B, C, D\}$ respectively are at least $1 + \epsilon$ distance apart.

Proof of Property 1. First, since $|AC| = |AG|$ and $|CD| = |GB|$, elementary geometry shows that the triangles ACD and AGB are identical. Thus $|AD| = |AB|$, angle $\angle ADC$ is an acute angle, and angle $\angle ACD$ is an obtuse angle. Then $\angle ACD > \angle ADC$ implies $|AD| > |AC|$, so

$$|AB| = |AD| > |AC| = 1 + \epsilon.$$

Second, since $|AF| = |GD| = 1$, we know that $AGDF$ is a parallelogram. This implies $|FD| = |AG|$ and $\angle FDB = \angle AGB = \angle ACD > \pi/2$ radians. Thus

$$|FB| > |FC| > |FD| = |AG| = 1 + \epsilon.$$

Third, since $\angle EDB > \angle FDB > \pi/2$ radians, we have

$$|EB| > |EC| > |ED| = |AB| > |AC| = 1 + \epsilon.$$

Therefore, any two vertices chosen from sets $\{A, E, F\}$ and $\{B, C, D\}$ respectively are at least $1 + \epsilon$ distance apart.

Property 2: The area of the parallelogram $ABDE$ is $\frac{1+\epsilon}{2}\sqrt{3 + 10\epsilon + 3\epsilon^2}$.

Proof of Property 2: By the Pythagoras theorem, the height of the parallelogram $ABDE$ is $\sqrt{|AC|^2 - (|CG|/2)^2}$. Thus the area of the parallelogram is

$$\begin{aligned} & |BD| \cdot \sqrt{|AC|^2 - \left(\frac{|CG|}{2}\right)^2} \\ = & (1+\epsilon) \cdot \sqrt{(1+\epsilon)^2 - \left(\frac{1-\epsilon}{2}\right)^2} \\ = & \frac{1+\epsilon}{2}\sqrt{3+10\epsilon+3\epsilon^2}. \end{aligned}$$

Property 3: $|BE| = \sqrt{3+7\epsilon+3\epsilon^2}$.

Proof of Property 3: Draw a line segment EH perpendicular to the line BD, as shown in Figure 1.10. Then $|EH|$ is the height of the parallelogram, and so

$$\begin{aligned} |EH| &= \sqrt{|AC|^2 - \left(\frac{|CG|}{2}\right)^2} \\ &= \frac{1}{2}\sqrt{3+10\epsilon+3\epsilon^2}. \end{aligned}$$

Applying the Pythagoras theorem to the right triangle BHE,

$$\begin{aligned} |BE| &= \sqrt{|EH|^2 + |BH|^2} \\ &= \sqrt{|EH|^2 + \left(|AE| + \frac{|BD|}{2}\right)^2} \\ &= \sqrt{\frac{3+10\epsilon+3\epsilon^2}{4} + \left(\frac{3(1+\epsilon)}{2}\right)^2} \\ &= \sqrt{3+7\epsilon+3\epsilon^2}. \end{aligned}$$

Now let P denote the parallelogram $ABDE$ with 6 points A, B, C, D, E, F on its boundary. This is our packing unit parallelogram. We use parallel packing to pack as many non-overlapping P's as possible into the circle $C(r)$, as shown in Figure 1.11. Let

$$l = \sqrt{3+7\epsilon+3\epsilon^2}.$$

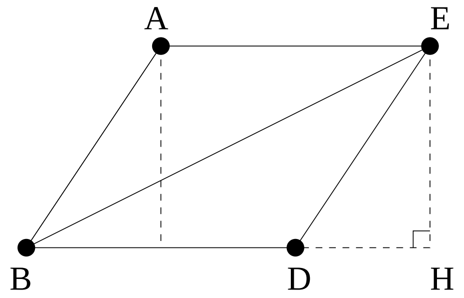

Figure 1.10: Computation of $|BE|$

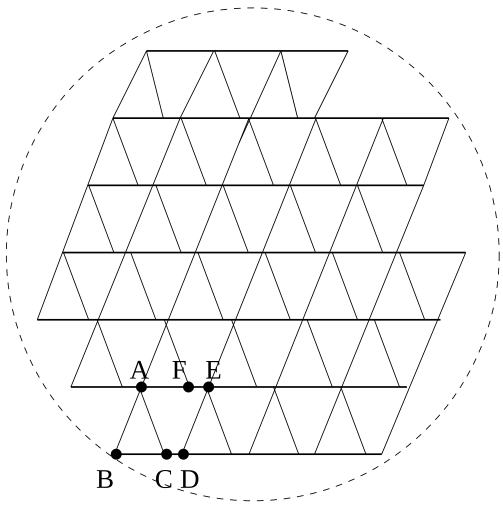

Figure 1.11: Parallel packing of parallelograms into a circle

By Property 3, any two points in P are at most l distance apart. Then the circle $C(r-l)$ concentric with $C(r)$ is entirely covered by P's because otherwise we could have packed one more P without reaching the boundary of $C(r)$. Thus, by Property 2, at least n parallelograms P can be packed into $C(r)$, where

$$\begin{aligned} n &\geq \frac{\text{Area of } C(r-l)}{\text{Area of } P} \\ &= \frac{\pi(r-l)^2}{(1+\epsilon)/2\sqrt{3+10\epsilon+3\epsilon^2}} \\ &= \frac{2\pi\left(r-\sqrt{3+7\epsilon+3\epsilon^2}\right)^2}{(1+\epsilon)\sqrt{3+10\epsilon+3\epsilon^2}}. \end{aligned}$$

Let V be the set of all points A, B, C, D, E, F in P's; that is, $V = \cup_P \{A, B, C, D, E, F\}$ and let $|V|$ denote the number of points in V. We have the following property:

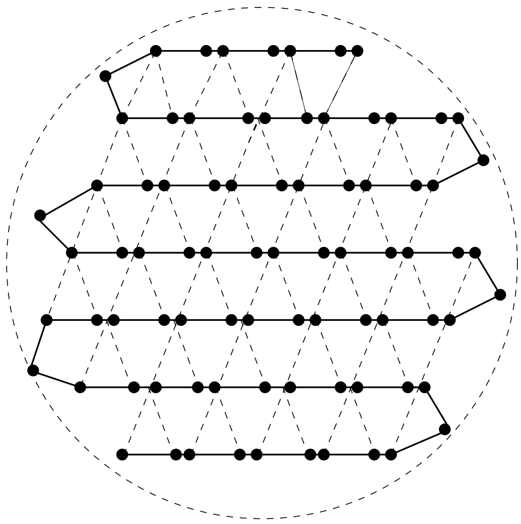

Figure 1.12: The construction of a path

Property 4: $|V| \geq 2n$.

Proof of Property 4: We may assume that the parallelograms are packed one by one into $C(r)$, from the lower layer to the higher layer, and in each layer from the left to the right. Since each time adding a parallelogram P increases $|V|$ by at least 2, Property 4 holds by induction.

Having the above assumptions and properties of parallelogram, now we construct a graph G within $C(r)$ as follows: let V be the vertex set of G and two vertices in V are connected if and only if they are at most 1 distance apart. By Property 0, vertices in each layer are connected to form a path; and by Property 1, any two vertices in different layers are not connected. So G is a union of disjoint paths. We can add some new vertices to G and let them be the intermediate vertices connecting paths of two adjacent layers, as shown in Figure 1.12. Since at least 1 new vertex is added to G, the final graph is a path with at least $|V|+1$ vertices. So it has a diameter of at least

$$|V| \geq 2n \geq \frac{4\pi\left(r - \sqrt{3 + 7\epsilon + 3\epsilon^2}\right)^2}{(1+\epsilon)\sqrt{3 + 10\epsilon + 3\epsilon^2}}.$$

Since the above argument holds for all the real numbers ϵ with $0 < \epsilon < 1$, we can make ϵ approaching 0. Since

$$\begin{aligned} & \lim_{\epsilon\to 0} \frac{4\pi\left(r-\sqrt{3+7\epsilon+3\epsilon^2}\right)^2}{(1+\epsilon)\sqrt{3+10\epsilon+3\epsilon^2}} \\ = & \frac{4\pi}{\sqrt{3}}\left(r-\sqrt{3}\right)^2 \\ = & \frac{4\pi r^2}{\sqrt{3}} + O(r), \end{aligned}$$

a graph with a diameter of $4\pi r^2/\sqrt{3} + O(r)$ always exists (by choosing ϵ very close to 0). Therefore our upper bound for the diameter (maximum necessary hop count) in Theorem 1,

$$\frac{4\pi}{\sqrt{3}}\left(r+\frac{1}{\sqrt{3}}\right)^2 - 1 = \frac{4\pi r^2}{\sqrt{3}} + O\left(r\right),$$

is very close to optimum with only a difference of $O(r)$.

1.7 Conclusion

In this paper, the maximum necessary number of hops needed to deliver a packet from a source to a destination has been found for a MANET within a circle of radius r, $r > 1$, assuming the transmission range of each mobile station is 1. Our proofs show that our result is very close to optimum with only a difference of $O(r)$.

References

[1] V.Bharghavan, A. Demers, S. Shenker, and L. Zhang, MACAW: A Medium Access Protocol for Wireless LANs, Proc. of SIGCOMM'94, 1994.

[2] X. Chen, J. Shen, and X. D. Jia, An Upper Bound for a Mobile Ad-Hoc Wireless Network, Proc. of the International Conference on Parallel and Distributed Processing Techniques and Applications, June 2001, pp. 1617-1620.

[3] C. L. Fullmer and J. J. Garcia-Luna-Aceves, Floor Acquisition Multiple Access (FAMA) for Packet-Radio Networks, Proc. of SIGCOMM'95, 1995.

[4] Web page of T.Hales www.math.lsa.umich.edu/˜heles/.

[5] C. R. Lin and M. Gerla, MACA/PR: An Asynchronous Multimedia Multihop Wireless Network, Proc. of INFOCOM'97, 1997.

[6] C. R. Lin and M. Gerla, Real-Time Support in Multihop wireless Network, ACM/Baltzer Wireless Networks, 5 (2) (1999).

[7] D. Surendran, The Conquest of the Kepler Conjecture, Math Horizons, 8 (2001), pp. 8-12.

[8] J. Wu and H. Li, A dominating-Set-Based Routing Scheme in Ad Hoc Wireless Networks, special issue on Wireless Networks in the Telecommunication Systems Journal, 3 (2001), pp. 63-84.